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Solving Pyraminx with Group Theory

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Abstract:

Pyraminx and Skewb are complex but not innovative or interesting puzzles developed from traditional puzzles. Although people have been able to solve these two kinds of Rubik's cube manually, the structure of the group of pyraminx and skewb still unclear and the algorithms for solving these two kinds of Rubik's cube through group theory are unknown. Firstly, this paper introduces the structures of pyraminx and skewb. Skewb is closely related to the pyraminx. Because they all have four axes of rotation. So they have related mathematical structures connecting with group theory. Secondly, in this paper, we expound on the adequate and important conditions for a solvable state and give group theoretical analyses and solutions for certain cases.

Keywords: Pyraminx, Group, Commutator

1. Introduction

The seminal pyraminx was invented by Uwe Mèffert in the year 1974, but it was not known by many until the wide-spread of the Rubik's Cube in 1981[1]. Meanwhile, the skewb, invented by Tony Durham, is a pyraminx shape-mod which was first called the Pyraminx Cube, working on the same four-axis mechanism. Pyraminx and skewb are different types of puzzles that have related mathematical structures connecting with group theory. Despite that both of them are popular as official WCA events, few studies have used practical mathematical methods to describe their group structure and solvability.

2. Background

2.1 Structure

The pyraminx is a tetrahedron, with 4 triangular faces which are all divided into 9 identical smaller triangles. We use the capital letters L, R, F, U denote single clockwise 120 degree turns of the Left, Right, Front, and Up corners, as shown in Fig.1, A pyraminx image labeled ABC. Besides, an apostrophe marks an anti-clock-wise turn.

A pyraminx has 4 faces, which consists of 4 corner pieces, 6 edge pieces, and 4 center pieces. Each corner piece has 3 visible faces (such as A), each edge piece has 2 visible faces (as C), and each centerpiece has single visible face (as B).

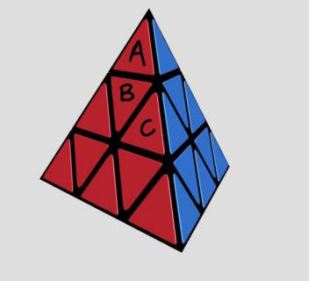

Figure 1. A pyraminx image labeled ABC

To name a corner piece, we simply use the lowercase letters to represent the tips, like l, r, t, f. Without regard to the orientation, the puzzle called as lrtf is

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same as ltrf. For convenience, we take the pieces as "oriented piece", which means lrtf are different from ltrf. As for the edge and center fragments, we are only naming the fragments available.

2.2 Moves

For pyraminx, each of the four corners is labeled as shown. The rotation of a single tip is just a simple turn for each corner because the 3 sides of the center pieces are linked together, so we won't use this kind of turn for algorithms. For example, a one-layer turn of the R corner, would be labeled as lowercase r (a 120° clockwise turn) or r' (anti-clockwise).

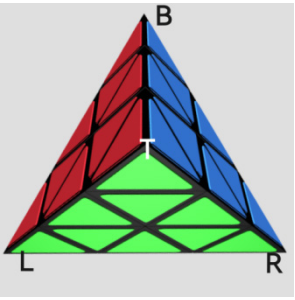

Figure 2. Top view of pyraminx

Likewise, two layers of clockwise turns at each corner will be marked as a capital letter like R, and the anti-clockwise like R'. And there are three basic types of pyraminx rotation operations, the first being the rotation of a single corner block, the exchange of two corner blocks and the cycle of three corner blocks.

2.3 Orientations.

Choosing red, green and blue three colors and finding the three corners that have specify color stickers. Then, orient the corners so that those stickers are all on the same face. Holding that face on the bottom for the rest of the solve. For adding the two edges, use moves of the form R D R' for the orientation 1 in clockwise turn, R D' R' for the orientation 2 in clockwise turn, R' D R for the orientation 1 in anti-clockwise, and R' D' R for the orientation -1 in anti-clockwise. Each move, with T turn, will bring one edge to the bottom layer. We could also define the orientation of a pyraminx in the similarly method. Suppose the entire face of the pyraminx cube is rotated clockwise, then the orientation of the cube is 1; if we need to rotate the pyraminx anticlockwise, then the orientation of the cube is -1.

2.4 Configurations.

For each center piece, we use the lowercase letter for its location. Next, we annotate them in $\{C_1, C_2, C_3, C_4\}$. The edge cubes are sorted in the same way. They denoted as $\{E_1, E_2, \cdots, E_6\}$. In every configuration, σ marks an arrangement of cubes with no directional corners, and τ is the similar meaning for cubes with unoriented edges.The orientations of the corner cubes are expressed as a vector $x = (x_1, x_2, x_3, x4)$, $x_k \in Z$, k = 1,2,3,4, representing the orientation of C_k. The orientations of all edge pieces are indicated as a vector $y = (y_1, y_2, \cdots, y_6)$, $y_i \in Z_2$, i = 1, 2, ...,6, representing the orientation of E_i. The total configuration is denoted as (σ, τ, x, y). The beginning solved configuration is denoted as $(\sigma_0, \tau_0,0,0)$.

For skewb, it's still all like the pyraminx pieces. For each center piece, we use the lowercase letter of its location. Next, we annotate them in $\{C_1, C_2, C_3, C_4\}$. The center pieces are sorted similarly and showed as$\{O_1, O_2, \cdots, O_6\}$. The rest can be done in the same manner.

3. Methodology

3.1 Pyraminx group definition and properties

Definition 3.1 (Pyraminx basic move). A pyraminx basic move is the rotation of a special entire corner of the tetrahedron in clockwise direction by $\frac{2\pi}{3}$.

We denote every move with same capital letter of corresponding entire corner. We list the set of these moves as $\{L, R, T, F\}$

where the move L rotates the entire corner of a tetrahedron with the label L by 2π/3 in a clockwise direction. The other moves work the equal way. We make the set of moves of the pyraminx into a group ($\mathbb{G}$, *). The group $\mathbb{G}$ is generated by all the moves of entire corners of the pyraminx. Moves are considered the same if they cause the same variation in the orientations. If M_1 and M_2 are moves in the group, M_1* M_2 means the move where we first do M_1 and then do M_2.

Theorem 3.1 A pyraminx group ($\mathbb{G}$, *) is built up with a set of moves of entire corners of the pyraminx.

$\mathbb{G} = ?L, R, T, F?$

Proof. 1. The $\mathbb{G}$ is closed because if M1 and M2 are moves in the group, M_1* M_2 is also the element in the group.

2.Let e be the empty move, M*e means first do M and then do nothing, so M*e=M, the $\mathbb{G}$ has identity element e. M is a basic move, then $M_3 = e$, and $M\text{-}_1 = M_2$. For compound move

$(M_1M_2\ldots M_{k-1}M_k)^{-1}=M_k^{-1}M_{k-1}^{-1}\ldots M_2^{-1}M_1^{-1}$

where M_i's are basic moves.

If M_1, M_2 belongs to $\mathbb{G}$ and C is an oriented cubit, M_i move C to the cubical Mi(C), therefore $(M_1*M_2)(C)=M_2(M_1(C))$.

Then, if M_3 also belongs to $\mathbb{G}$ $[(M_1*M_2)*M_3]$ $(C)=M_3(M_2(M_1(C)))=[M_1*(M_2*M_3)]$ (C). So $M_1*(M_2*M_3)$ $=(M_1*M_2)*M_3$. The operation is associative.

Hence, ($\mathbb{G}$, *) is truly a group.[2]

Since tips are irrelevant to the solution, we named each edge piece by its initial place in different $2\times2\times2$ pyraminx with two lowercase letters from (l, r, t, f) like lr so that we could write every movement of the pyraminx by using a modified cycle notation, and it describes the direction in which each oriented cube move.

For example,

$F\to(\backslash genfrac0\, ptltftlfrfrtlrltrfftlfrtlr)$

Proposition 3.1. The pyraminx group ($\mathbb{G}$, *) is non-Abelian.

Proof. The move LR is not equal to RL, so the space $\mathbb{G}$ is non-Abelian.

We can define a map ϕ: $\mathbb{G}\to A_6$ as follows: there is a set of moves M in $\mathbb{G}$ that rearranges the edge cubits, it defines a permutation of the 6 unoriented edge pieces. The permutation of 4 unoriented corner cubits is fixed.

Then, to show how every move affects the orientation of the cubits, we can use map $\nu:\mathbb{G}\to\mathbb{Z}_3^4$ that represent the orientation of the corner cubits and map $\omega:\mathbb{G}\to\mathbb{Z}_2^6$ that represent the orientation of the edge cubits. Besides, all the basic moves produce a 3-cycle, so only even permutations are given.

3.2 Generators Construction and group structure

Lemma 3.1 There is an edge 3-cycle move in $M\in\mathbb{G}$, $M=(C_1,C_2,C_3)$, as C_1,C_2,C_3 are 3 unoriented edge pieces on same face.

Proof. As shown is Fig.2 Top view of pyraminx. Using $[\alpha,\beta]=\alpha\beta\alpha^{-1}\beta^{-1}$ as the commutator of $\alpha and\beta$. If $\alpha,\beta\in S_6$, let $\alpha=L,\beta=F$, then the support of α and the support of β has one edge cubit labeled as 3,. Thereforethe commutator is the 3-cycle

$[\alpha,\beta]=(lt,\ ft,lf)$

Lemma 3.2 There is an edge 2-flip move in $M\in\mathbb{G}$, it is an orientation switch which only a pair of edge pieces (C_1, C_2) is twisted and the permutation is not changed. Using $[\alpha,\beta]=\alpha\beta\alpha^{-1}\beta^{-1}$ as the commutator of $\alpha and\beta$. The $[\alpha,\beta][\alpha\ ¡®,\theta]$ means first act the commutator of $\alpha and\beta$, then use the commutator of $\alpha\ ¡®$ and θ. If $\alpha,\beta,\theta\in S_6$, let $\alpha=F,\alpha\ ¡®=F',\beta=L',\theta=U$. We use different order of letters to represent the orientations of the edge pieces, then we can represent the orientation by choosing a number from each edge pieces.

$[\alpha,\beta][\alpha\ ¡®,\theta]=(lt,tl)(ft,tf)$

Overall, the pyraminx group has the structure [3]

$\mathbb{G}\to\left[Z_2^6?A_6\right]\times Z_3^4$

3.3 Skewb group definition and properties.

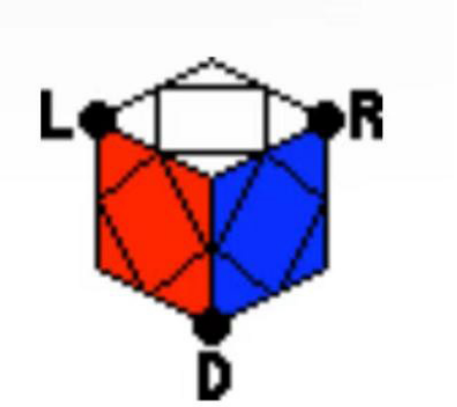

Figure 3. Mark the skewb of LRD

The Skewb is a cube, as shown in Fig.3, with 6 faces each divided into 1 center square and 4 corner isosceles right triangles. We use letters L, R, D, and B to denote single 120-degree clockwise rotations around 4 different vertexes as shown in Fig. 1. A pyraminx image labeled ABC. and a letter with <'> denotes 120-degree anticlockwise.

A skewb has 6 faces, which consists of 8 corners and 6 square center pieces. Each corner piece has 3 visible faces, and each center square has single visible faces. We use similar method to name the pieces of the skewb, because skewb is closely related to the pyraminx due to the same number of rotational axes,which are both 4.[1] The six marginal sections of the pyramid correspond to the six square faces of Skewb, and the four angular sections (without tips) correspond to the four corners of Skewb.

Both Skewb and Pyraminx have four axes of rotation, so their group properties are almost the same. The only difference is that we denote the set of moves as

$\{L,R,D,B\}$

Through theorem3.1, we also get that a skewb group ($\mathbb{G}$, *) is indeed a group, but it is non-Abelian.

4. Modelling

Implementing a representation of a Rubik's Cube's state, rotation functions, and group operations in Python involves defining a suitable data structure for the cube's state and then writing functions to manipulate that state.

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Below is a simplified version focusing on a 3x3x3 Pyraminx for brevity and clarity.

4.1 Data Structure for Cube State

For a 3x3x3 Pyraminx, we can represent the state as a 3D array (or a list of lists of lists) where each inner list contains the colors of the stickers on that face, organized by layer and row. However, for simplicity and efficiency, we'll use a flat list and some indexing logic to simulate the 3D structure.

4.2 Two different codes

Here are two simple python codes for solving the Rubik's Pyramid, divided into code1 and code2. And Code 2 is a condensed version of code 1.Let's start with code 1.

```
ef generate_3_tower_magic_cube(state=None):
    if state is None:
        state = [1, 2, 3, 4, 5, 6]
    if len(state) == 6:
        # Check whether conditions are met
        if is_valid_3_tower_magic_cube(state):
            return [state]
        else:
            return []
    else:
        # Try to place numbers for each position
        solutions = []
        for i in range(6):
            if i not in state:
                new_state = state[:]
                new_state.append(i + 1)
                solutions.extend(generate_3_tower_magic_c
        return solutions
```

```

def is_valid_3_tower_magic_cube(state):
    # Check that each layer is in order
    for i in range(3):
        if state[i] + 3 != state[i + 3]:
            return False
    # Check that the middle layer is symmetrical
    if state[3] != state[4]:
        return False
    return True

# Generates all possible third order pyramid cube states
solutions = generate_3_tower_magic_cube()
print(solutions)
```

This code 1 looks a little hard to understand, but the steps are clearly delineated. Let's have a little solution, first check whether the conditions meet the third-order pyramid cube, and then give each position name a number, and then start to calculate the end of the operation to see the order of each layer, if the order is equal, then the result is not established, and then back, because false means that the conditions are not met, If the next step to check if the middle layer is symmetric is also false, it will go back again until it becomes return true.

Unfortunately, code 1 doesn't work, but code 2 does, so let's take a look at code 2.

```
def generate_3_tower_magic_cube(state=None):
    if state is None:
        state = [1, 2, 3, 4, 5, 6, 7, 8, 9]
    if len(state) == 1:
        return [[state]]
    solutions = []
    for i in range(len(state)):
        for sub_solution in generate_3_tower_magic_cube(
            solutions.append([state[i]] + sub_solution)
    return solutions

# Use example
solutions = generate_3_tower_magic_cube()
print(f"Total number of solutions: {len(solutions)}")
# Print the first few solutions
for i, solution in enumerate(solutions[:5]):
    print(f"Solution {i+1}: {solution}")
```

This is code 2, which is cleaner and more condensed than code 1, and it's actually pretty much the same.Fortunately, this code has results.

```
Total number of solutions: 362880
Solution 1: [1, 2, 3, 4, 5, 6, 7, 8, [9]]
Solution 2: [1, 2, 3, 4, 5, 6, 7, 9, [8]]
Solution 3: [1, 2, 3, 4, 5, 6, 8, 7, [9]]
Solution 4: [1, 2, 3, 4, 5, 6, 8, 9, [7]]
Solution 5: [1, 2, 3, 4, 5, 6, 9, 7, [8]]
```

In order to better understand how the Rubik's Pyramid cube is solved, let's take a simple and easy to understand example.

4.3 Solving the tips and centers

The solution for pyraminx is to start with the four corners and turn them so that they match the center block. In Figure 4, for example, rotate the top corner, let's call this corner B, rotate B to the correct side, which means that this side is in the same color, all yellow, and then rotate the five corners in turn, all to the correct side.

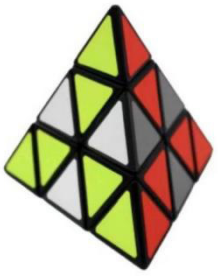


Figure 4. A pyraminx with different colored blocks

4.4 Two edges next to one corner

After turning to the same side, begin to solve the three sides around the top corner. The fragments with * in the middle of both sides do not move, stay in place, and the left and right algorithm is to address this issue right now(-see Figure 5).

Eg. For left algorithm:

LF L' F '↦(1 2 3 4 5 6¦5 2 1 4 6 3)

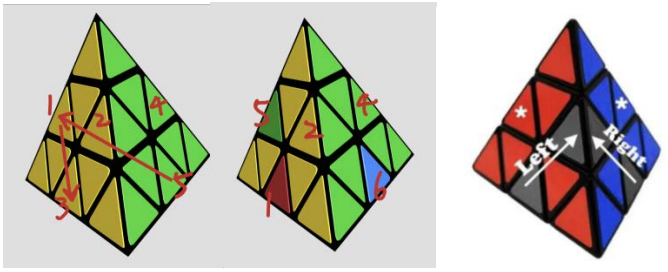
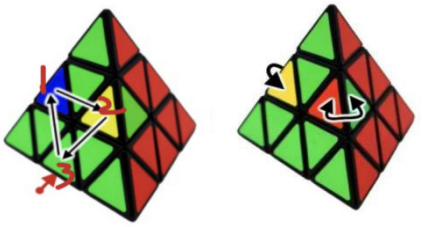


Figure 5. diagrams of the transition process

4.5 Last layer edges

4.5.1 edge 3-cycle

All that's left is to permute/cycle the last layer edges.
Eg. For clockwise cycle, as shown in Figure 6(a):
LF L' F '↦(1 2 3 4 5 6¦2 3 1 4 5 6)



(a) **(b)**

Figure 6. diagrams of the transition process

4.5.2 edge 2-flip

All that's left is to permute/cycle the last layer edges.
This is orientation switch, but σ is not injective.
As shown in Figure 6(b): F'UFU'FL'F'L↦ (1 2 3 4 5 6¦1 2 3 4 5 6)
This is an example of one basic solution for pyraminx, but there are many different ones we haven't listed here.

5. Data

Research on Rubik's cube is mainly divided into two directions: human restoration algorithm and computer restoration algorithm. Although these two kinds of algorithms are very different, they are both based on the basic principles of group theory. This shows that the computer's recovery of the Rubik's cube algorithm is achieved by applying the same techniques of group theory that are applied to the process of dealing with the pyramid's cube group, in which the principle of group theory and the data structure of multidimensional array are the key components.

6. Discussion

The relationship between pyramid groups and group theory is mainly reflected in mathematical abstraction and the analysis of geometric structures. Its complex geometry and symmetries can provide a concrete example for group theory to study its symmetries and transformations. Their high degree of symmetry can be described by concepts in group theory. For example, we treat each face of a pyramid as an element, and these elements form a group under certain operations, such as rotation and reflection. This group is called a permutation group or symmetry group, and it contains all possible symmetry operations such that the pyramid remains unchanged. These operations form a finite group because their number is finite and these operations can be combined into new operations. For example, two rotations can be combined into one rotation, which conforms to the closure of the group. In addition, every operation has an inverse, that is, the original state can be restored by the opposite operation, which corresponds to the existence of the identity element.

7. Conclusion

In this article, we try to use group theory to explain the reduction steps of the pyraminx cube and find the structure of the pyraminx and skewb group. We found that these two seemingly unrelated Rubik's cubes are actually very similar in structure and the Mathematical explanation of their solving method is more complex and still needs to be improved.

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